EE 232 Lightwave Devices
Lecture 13: Rate Equations and Dynamic Response of Semiconductor Lasers

Reading: Chuang, Sec. 11.1 – 11.2
(See also Coldren, Sec. 5.1 – 5.3)
(The Notes follow primarily Coldren’s book)

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Rate Equations

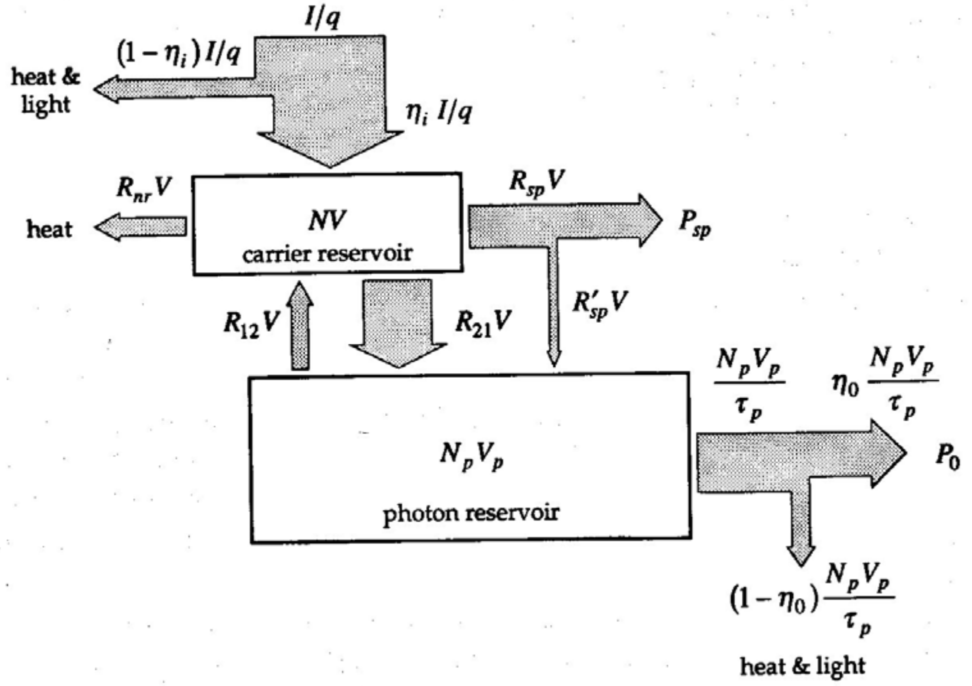


FIGURE 5.1 Model used in the rate equation analysis of semiconductor lasers.

Coldren, p. 186

Notation in Coldren's Book

$$N_p \leftrightarrow S$$

$$R_{sp} \leftrightarrow BN^2$$

$$R_{nr} \leftrightarrow AN + CN^3$$

Rate Equations:

$$\begin{cases} \dfrac{dN}{dt} = \dfrac{\eta_i I}{qV} - \dfrac{N}{\tau} - v_g g(N)S \\ \dfrac{dS}{dt} = \Gamma v_g g(N)S - \dfrac{S}{\tau_p} + \Gamma\beta R_{sp} \end{cases}$$

Note that

$$\frac{N}{\tau} = AN + BN^2 + CN^3$$

Simplified Analysis: Steady State

$R_{sp} \approx 0$

Steady State Solution:

$$\frac{dS}{dt} = 0 = \Gamma v_g g(N) S - \frac{S}{\tau_p}$$

$$\Rightarrow g(N) = \frac{1}{\Gamma v_g \tau_p} = \frac{\alpha_{total}}{\Gamma} = g_{th}$$

$\Rightarrow$ Gain "clamped" at threshold

$$\frac{dN}{dt} = 0 = \frac{\eta_i I}{qV} - \frac{N}{\tau} - v_g g(N) S$$

(1) Below Threshold: $S = 0$

$$I = \frac{qNV}{\eta_i \tau}$$

(2) Above Threshold: $S > 0$

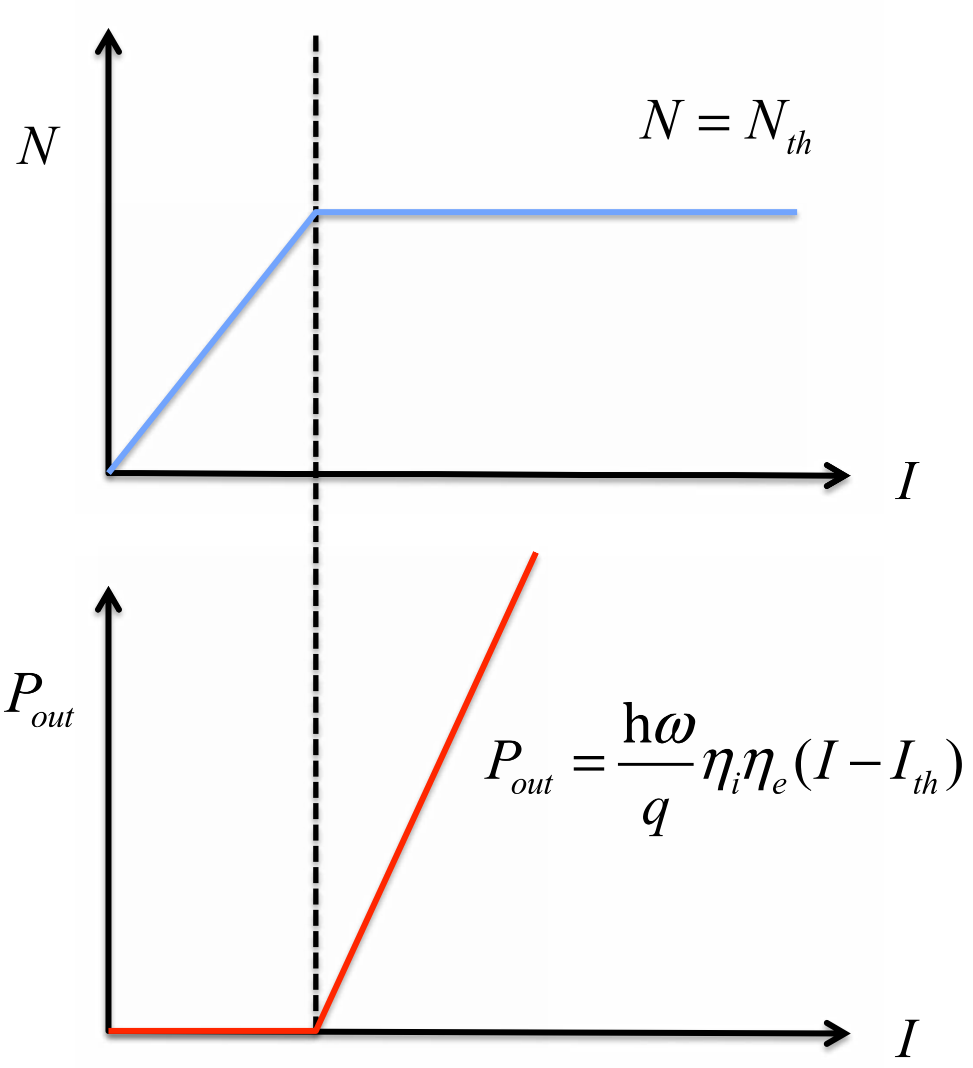
$$S = \frac{1}{v_g g_{th}} \frac{\eta_i}{qV} (I - I_{th})$$

$$P_{out} = \frac{\hbar\omega}{\tau_p} \frac{\alpha_m}{\alpha_i + \alpha_m} \frac{VS}{\Gamma}$$

$$= \frac{1}{v_g g_{th}} \frac{\hbar\omega}{\tau_p} \frac{\alpha_m}{\alpha_i + \alpha_m} \frac{\eta_i}{qV} \frac{V}{\Gamma} (I - I_{th})$$

$$P_{out} = \frac{\hbar\omega}{q} \eta_i \eta_e (I - I_{th})$$

Simplified Analysis: Steady State



$$N = N_{th}$$

$$P_{out} = \frac{h\omega}{q}\eta_i\eta_e(I - I_{th})$$

Detailed Analysis: Steady State Solutions

$$\frac{dS}{dt} = 0 = \Gamma v_g g(N) S - \frac{S}{\tau_p} + \Gamma \beta R_{sp}$$

$$\Rightarrow S(N) = \frac{\Gamma \beta R_{sp}(N)}{\frac{1}{\tau_p} - \Gamma v_g g(N)} \qquad (1)$$

$$\frac{dN}{dt} = 0 = \frac{\eta_i I}{qV} - \frac{N}{\tau(N)} - v_g g(N) S(N)$$

$$\Rightarrow I(N) = \frac{qV}{\eta_i}\left(\frac{N}{\tau(N)} + v_g g(N) S(N)\right) \qquad (2)$$

Equations (1) and (2) are both functions of carrier concentration N Use N as indepdent parameter, one can calculate the L-I curve by calculate $S(N)$ and $I(N)$. N ranges from 0 to N_{th}

Detailed Analysis: Steady State Solutions (Cont'd)

Below Threshold

$$S(N) \approx 0$$

Output power: $P_0 \approx 0$

$$I(N) = \frac{qV}{\eta_i}\left(\frac{N}{\tau(N)}\right)$$

Above Threshold

$$N \rightarrow N_{th} \ , \quad g_{th} = g(N_{th}) = \frac{\alpha_i + \alpha_m}{\Gamma}$$

$$\frac{1}{\tau_p} = v_g \cdot (\alpha_i + \alpha_m) = \Gamma v_g g_{th}$$

$$\Rightarrow S(N) = \frac{\beta R_{sp}(N) / v_g}{g_{th} - g(N)}$$

$$I(N) = \frac{qV}{\eta_i}\left(\frac{N_{th}}{\tau(N_{th})}\right) + \frac{qV}{\eta_i}\left(v_g g(N) S(N)\right)$$

$$= I_{th} + \frac{qV}{\eta_i}\left(v_g g(N) S(N)\right)$$

$$P_o = \eta_i \eta_0 \frac{\mathrm{h}\omega}{q}(I - I_{th})$$

Steady State (DC) Solutions

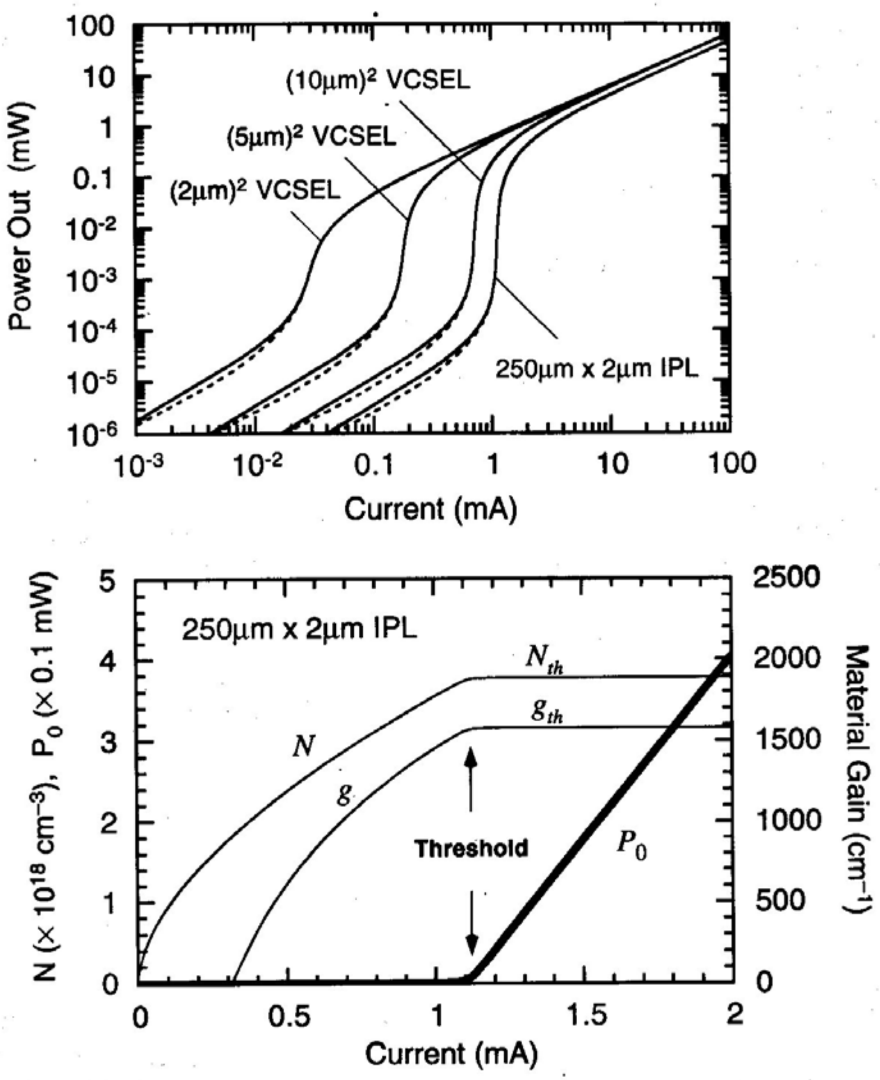


FIGURE 5.2 Upper plot: light vs. current in two different lateral-size 3-QW VCSELs and in a single QW in-plane laser (IPL) (all lasers use InGaAs/GaAs 80Å QWs). Lower plot: Light vs. current on a linear scale for the same in-plane laser. Plot also shows carrier density and material gain vs. current.

- L-I curve in log-log plot
  - Threshold current shows up as "shoulder"
  - Spontaneous emission is clearly seen below threshold
- L-I curve in linear scale
  - Gain is "clamped" at threshold
  - Carrier concentration is also clamped at threshold

Differential Analysis

$$d\left[\frac{dN}{dt}\right]=\frac{\eta_i}{qV}dI-\frac{dN}{\tau_{\Delta N}}-\left(v_g g dS+v_g S dg\right)$$

$$d\left[\frac{dS}{dt}\right]=\Gamma\left(v_g g dS+v_g S dg\right)-\frac{dS}{\tau_p}+\Gamma\frac{dN}{\tau_{\Delta N}^{'}}$$

$$\frac{N}{\tau(N)}=AN+BN^2+CN^3$$

$$\frac{1}{\tau_{\Delta N}}=A+2BN+3CN^2$$

$$\frac{1}{\tau_{\Delta N}^{'}}=\frac{d}{dN}(\beta BN^2)=2\beta BN+\frac{d\beta}{dN}BN^2$$

Gain is generally both a function of N and S:

$$g=g(N,S)$$

$$dg=\frac{\partial g}{\partial N}dN+\frac{\partial g}{\partial S}dS=a\cdot dN-a_p dS$$

Example:

Logarithmic gain model:

$$g(N,S)=\frac{g_0}{1+\varepsilon S}\ln\left(\frac{N}{N_{tr}}\right)$$

$$a=\frac{\partial g}{\partial N}=\frac{g_0}{1+\varepsilon S}\frac{1}{N}$$

$$a_p=-\frac{\partial g}{\partial S}=\frac{\varepsilon g}{1+\varepsilon S}$$

Differential Analysis

The differential rate equations can be expressed as

$$\frac{d}{dt}\begin{bmatrix} dN \\ dS \end{bmatrix} = \begin{bmatrix} -\gamma_{NN} & -\gamma_{NS} \\ \gamma_{SN} & -\gamma_{SS} \end{bmatrix} \begin{bmatrix} dN \\ dS \end{bmatrix} + \frac{\eta_i}{qV}\begin{bmatrix} dI \\ 0 \end{bmatrix}$$

$$\begin{cases} \gamma_{NN} = \dfrac{1}{\tau_{\Delta N}} + v_g a S \\ \gamma_{NS} = v_g g - a_p v_g S \\ \gamma_{SN} = \Gamma v_g S a \\ \gamma_{SS} = \Gamma v_g S a_p \end{cases}$$

Small-Signal Analysis

$$dI(t) = I_1 e^{j\omega t};\ dN(t) = N_1 e^{j\omega t};\ dS(t) = S_1 e^{j\omega t}$$

$$\begin{bmatrix} j\omega + \gamma_{NN} & \gamma_{NS} \\ -\gamma_{SN} & j\omega + \gamma_{SS} \end{bmatrix} \begin{bmatrix} N_1 \\ S_1 \end{bmatrix} = \frac{\eta_i I_1}{qV} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Delta \equiv \begin{vmatrix} j\omega + \gamma_{NN} & \gamma_{NS} \\ -\gamma_{SN} & j\omega + \gamma_{SS} \end{vmatrix}$$

$$= \left(\gamma_{NS}\gamma_{SN} + \gamma_{NN}\gamma_{SS}\right) - \omega^2 + j\omega\left(\gamma_{NN} + \gamma_{SS}\right)$$

$$= \omega_R^2 - \omega^2 + j\omega\gamma$$

$$N_1 = \frac{\eta_i I_1}{qV} \frac{\gamma_{SS} + j\omega}{\omega_R^2} H(\omega)$$

$$S_1 = \frac{\eta_i I_1}{qV} \frac{\gamma_{SN}}{\omega_R^2} H(\omega)$$

$$\boxed{H(\omega) = \frac{\omega_R^2}{\Delta} = \frac{\omega_R^2}{\omega_R^2 - \omega^2 + j\omega\gamma}}$$

$$\omega_R^2 = \gamma_{NS}\gamma_{SN} + \gamma_{NN}\gamma_{SS}$$

$$\boxed{\omega_R^2 \approx \frac{v_g a S}{\tau_p}} \quad \text{(above threshold)}$$

$$\gamma = \gamma_{NN} + \gamma_{SS}$$

$$\gamma = v_g a S\left(1 + \frac{\Gamma a_p}{a}\right) + \frac{1}{\tau_{\Delta N}} + \frac{\Gamma R'_{sp}}{S}$$

$$\boxed{\begin{aligned} &\gamma = K f_R^2 + \gamma_0 \\ &K = 4\pi^2 \tau_p \left(1 + \frac{\Gamma a_p}{a}\right) \\ &\gamma_0 \approx \frac{1}{\tau_{\Delta N}} \end{aligned}}$$

Frequency Response

Frequency response of the laser:

$$\frac{S_1}{I_1} = \frac{\eta_i}{qV}\frac{\gamma_{SN}}{\omega_R^2}H(\omega)$$

$$P_1 = \eta_0 \mathrm{h}\omega V v_g g_{th} S_1$$

$$\boxed{\frac{P_1}{I_1} = \underbrace{\left(\eta_i\eta_0\frac{\mathrm{h}\omega}{q}\right)}_{\text{DC Quantum Efficiency}}\underbrace{H(\omega)}_{\text{Normalized Frequency Response}}}$$

$$\boxed{f_R = \frac{1}{2\pi}\omega_R = \frac{1}{2\pi}\sqrt{\frac{v_g a S}{\tau_p}}}$$

f_{3-dB} can be obtained by solving

$$\left|H(\omega)\right|^2 = \frac{1}{2}$$

$$f_{3-dB} = f_R\sqrt{1+\sqrt{2}} \approx 1.55 f_R$$

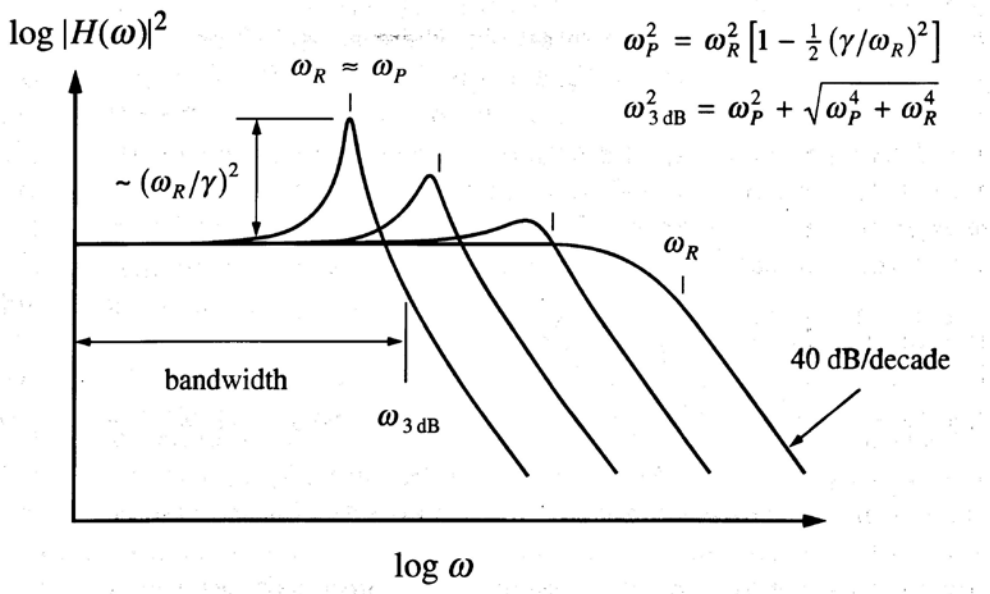


FIGURE 5.4 Sketch of the modulation transfer function for increasing values of relaxation resonance frequency and damping factor, including relationships between the peak frequency, ω_P, the resonance frequency, ω_R, and the 3 dB down cutoff frequency, $\omega_{3\,\mathrm{dB}}$.

Typical values:

$v_g \sim 10^{10}$ cm/sec, $a \sim 10^{-16}$ cm^2, $\tau_p \sim 1ps$

$S \sim 10^{15}$ 1/cm^3

$$f_R = \frac{1}{2\pi}\omega_R = \frac{1}{2\pi}\sqrt{\frac{v_g a S}{\tau_p}} \sim 5GHz$$

Small-Signal Analysis

3dB frequency:

$$\left|H(\omega_{3dB})\right|^2 = \frac{1}{2}$$

$$\left(\omega_R^2 - \omega_{3dB}^2\right)^2 + \left(\omega_{3dB}\gamma\right)^2 = 2\omega_R^4$$

$$\omega_{3dB}^4 + (\gamma^2 - 2\omega_R^2)\omega_{3dB}^2 - \omega_R^4 = 0$$

$$\omega_{3dB}^2 = \omega_p^2 + \sqrt{\omega_p^4 + \omega_R^4}$$

$$\omega_p^2 = \frac{-(\gamma^2 - 2\omega_R^2)}{2} = \omega_R^2 - \frac{1}{2}\gamma^2$$

Note

$$\gamma^2 = \left(Kf_R^2\right)^2 = \left(\frac{K\omega_R^2}{4\pi^2}\right)^2$$

At high bias, $\omega_p^2 >> \omega_R^2$

$$\omega_{3dB}^2 \approx \omega_p^2 + \omega_R^2 = 2\omega_R^2 - \frac{1}{2}\gamma^2$$

Maximum 3dB bandwidth occurs when

$$\frac{d(\omega_{3dB}^2)}{d(\omega_R^2)} = 0$$

After some algebra, you can find

$$f_{R,\max} = \frac{2\pi\sqrt{2}}{K}$$

$$\boxed{f_{3-dB,\max} = f_{R,\max} = \frac{2\pi\sqrt{2}}{K}}$$

Fundamental limit of direct modulation